Weyl functions of Dirac systems and of their generalizations: integral representation, inverse problem, and discrete interpolation

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Abstract

Self-adjoint Dirac systems and subclasses of canonical systems, which generalize Dirac systems are studied. Explicit and global solutions of direct and inverse problems are obtained. A local Borg-Marchenko-type theorem, integral representation of the Weyl function, and results on interpolation of Weyl functions are also derived.

1 Introduction

Weyl functions (also called Weyl-Titchmarsh or M-functions) and their generalizations are an important and much used tool in the spectral theory of differential equations (see a necessarily small part of recent papers and books on this topic and various references therein [3–5, 10, 11, 14, 16–18, 20, 32, 37, 41, 49, 51]). Following the seminal work [27] and more general constructions in [45, 49] (see also some references therein), one can use structured operators to solve inverse problems for Krein, Dirac, canonical, non-classical and non-self-adjoint systems and their discrete analogues (see, for instance, [2, 15, 16, 37, 40–42]). It was proved in [16, 36, 37, 40–42] that the kernels of these structured operators are connected with the Weyl functions via

some kinds of Fourier transformations and can be recovered directly from the Weyl functions. It is also true that the Weyl functions can be represented as Fourier transformations of the corresponding kernels of the structured operators. Important related works [18, 50] on the A-amplitudes for Schrödinger operators gave rise to a whole series of interesting papers and results on the high energy asymptotics of Weyl functions and local Borg-Marchenko-type uniqueness theorems (see, for instance, [9, 11, 12, 30] and references therein). See also further discussion of A-amplitudes in [5]. Finally, the interpolation of Weyl functions for scalar Schrödinger operators using the values of Weyl functions on a discrete countable set was dealt with in two interesting papers [6, 35]. The second and more general paper [35] is based on the A-amplitude representation of Weyl functions.

In this paper we consider self-adjoint Dirac systems and subclasses of canonical systems, which generalize Dirac systems. We consider direct and inverse problems, integral representations of Weyl functions, and interpolation of Weyl functions. Connections with Schur coefficients and A-amplitude are discussed.

In Section 2 we formulate some previous results for Dirac systems, prove formula (2.21), which solves the inverse problem in a more general setting than previously considered, and derive for Dirac systems analogs of some A-amplitude-type results for Schrödinger equations. In Section 3 subclasses of canonical systems are studied and the case of rational Weyl matrix functions is discussed. Explicit solutions of direct and inverse problems are given for that case.

A general case, where Weyl functions are not necessarily rational, is considered in Section 4. An integral representation of the Weyl function, a solution of the inverse problem for generalizations of Dirac systems directly via Weyl functions, and a local Borg-Marchenko-type theorem are obtained there. Section 5 is dedicated to interpolation of Weyl functions.

As usual, by $\mathbb{C}$ we denote the complex plane, by $\mathbb{C}_+$ we denote the open upper semi-plane, $\mathbb{R}$ is the real axis, $\mathbb{R}_+$ is the positive real axis, $\mathbb{N}$ is the set of positive integers, and $\mathbb{N}_0$ is the set of non-negative integers. We have $\beta' = \frac{d}{dx}\beta$. By $L^2_{p \times p}(-\infty, \infty)$ we mean the space of $p \times p$ matrix functions with entries which belong to $L^2(-\infty, \infty)$. The $p \times p$ identity matrix is denoted

by I_p, the identity operator is denoted by I, and diag denotes a diagonal or block diagonal matrix. The set of bounded linear operators acting from H_1 to H_2 is denoted by $\{H_1, H_2\}$ and σ is used to denote the spectrum.

2 Self-adjoint Dirac system: A-amplitude and inverse problem

Consider the matrix self-adjoint Dirac system

$$\frac{d}{dx}u(x,z) = i\big(zj + jV(x)\big)u(x,z), \quad x \geq 0, \tag{2.1}$$

where

$$j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \tag{2.2}$$

I_p is the $p \times p$ identity matrix, V is an $m \times m$ $(m = 2p)$ matrix function with the $p \times p$ block entry v (often referred to as the potential) being locally summable. If v is summable on each finite interval $[0, l]$, that is, if v is locally summable, there is (see Proposition 5.2 and its proof in [40]) a unique $p \times p$ matrix function $\varphi(z)$ such that

$$\int_0^\infty \begin{bmatrix} I_p & i\varphi(z)^* \end{bmatrix} \widehat{u}(x,z)^*\widehat{u}(x,z) \begin{bmatrix} I_p \\ -i\varphi(z) \end{bmatrix} dx < \infty \quad (z \in \mathbb{C}_+), \tag{2.3}$$

where $\widehat{u}$ is the fundamental solution of (2.1) normalized by the condition

$$\widehat{u}(0,z) = K^*, \quad K := \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix}. \tag{2.4}$$

Definition 2.1 *The $p \times p$ matrix function $\varphi(z)$ $(z \in \mathbb{C}_+)$, which satisfies (2.3), is called the Weyl function of system (2.1).*

The Weyl function of (2.1) is holomorphic in $\mathbb{C}_+$ and $\Im\varphi \geq 0$, which we discuss in detail in Section 5.

We always assume that v is measurable. When v is also locally bounded, one can recover it from the spectral function or directly from the Weyl function using Theorems 4.2 or 5.4 from [40], respectively. To recover v from the

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Weyl function one should notice that the Weyl function of system (2.1) on the semi-axis is a so called Weyl point and, according to Proposition 5.2 and formula (5.3) from [40], Theorem 5.4 from [40] is applicable.

Let us describe the procedure suggested in Theorem 5.4 [40] to recover system on any intervals. For this purpose we need a family of operators which are associated with system (2.1) and have difference kernels:

$$S_l = \frac{d}{dx} \int_0^l s(x-t) \cdot dt, \quad s(x) = -s(-x)^*,$$

where

$$s(x) = \left(\frac{d}{dx} \frac{i}{4\pi} e^{\eta x} \text{l.i.m.}_{a\to\infty} \int_{-a}^{a} e^{-i\zeta x} (\zeta + i\eta)^{-2} \varphi(\zeta + i\eta) d\zeta \right)^*, \quad \eta > 0.$$

(2.5)

Here $(\zeta + i\eta)^{-2} \varphi(\zeta + i\eta) \in L^2_{p\times p}(-\infty, \infty)$ for every fixed $\eta > 0$, l.i.m. denotes the entrywise limit in the norm of $L^2(0, l)$ ($l > 0$), and l.i.m. in the right-hand side of (2.5) is differentiable. Moreover, $s(x)$ is boundedly differentiable, $s(0) = \frac{1}{2} I_p$, and $S_l > 0$ for all $l > 0$. Thus, we have

$$S_l = I + \int_0^l k(x-t) \cdot dt > 0, \quad k(x) := s'(x) \ \left(s' = \frac{d}{dx} s\right), \quad k(x) = k(-x)^*,$$

(2.6)

and S_l, $S_l^{-1} \in \{L^2_p(0, l), L^2_p(0, l)\}$. Notice, that for convenience purposes φ, s, and S in the present paper differ by scalar constant factors from the objects denoted by the same letters in [40].

We now introduce the matrix functions

$$\theta_1(x) = [I_p \ \ 0]\widehat{u}(x, 0), \quad \theta_2(x) = [0 \ \ I_p]\widehat{u}(x, 0). \tag{2.7}$$

According to formula (4.16) from [40] we have

$$\theta_2(x) = \frac{1}{\sqrt{2}} \left([-I_p \ \ I_p] - \int_0^{2x} k(t)^* S_{2x}^{-1}[2s(t) \ \ I_p]dt \right), \tag{2.8}$$

where S_r^{-1} ($r = 2x$) is applied to $[2s(t) \ \ I_p]$ columnwise. It easily follows from (2.1) and (2.4) that

$$\widehat{u}(x, 0)^* j \widehat{u}(x, 0) \equiv J, \quad \widehat{u}(x, 0) J \widehat{u}(x, 0)^* \equiv j, \quad J := \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}. \tag{2.9}$$

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Using (2.1), (2.7), and the second relation in (2.9) we obtain

$$v(x) = i\theta_1'(x)J\theta_2(x)^*. \tag{2.10}$$

Finally, in view of (2.1), (2.4), (2.7), and the second relation in (2.9) we get equalities

$$\theta_1(0) = \frac{1}{\sqrt{2}}[I_p \quad I_p], \quad \theta_1(x)J\theta_2(x)^* \equiv 0, \quad \theta_1'(x)J\theta_1(x)^* \equiv 0, \tag{2.11}$$

which uniquely determine θ_1 assuming that θ_2 is already given. Rewrite Theorem 5.4 from [40] in a slightly modified form.

Theorem 2.2 *[40] Let φ be the Weyl function of the matrix Dirac system (2.1), where the potential v is locally bounded, that is,*

$$\sup_{0<x<l} \|v(x)\| < \infty \quad \text{for any} \quad l > 0. \tag{2.12}$$

Then v is uniquely recovered from φ by the formulas (2.10), (2.8), and (2.11), whereas s and S_{2x} in (2.8) are given by (2.5) and (2.6), respectively.

Remark 2.3 *Notice that according to formula (3.26) from [40] we have*

$$\varphi(z) = 2z \int_0^\infty e^{izx} s(x)^* dx, \tag{2.13}$$

that is, the kernel s, which is used in Theorem 2.2 to solve the inverse problem, is a Dirac system's analog of the A-amplitude. (To be more precise, $k = s'$ is the analog.) Following M.G. Krein, the matrix functon k is called an accelerant in [2].

Clearly, formulas (2.5) and (2.13) are closely related (and (2.13) was used in [40] to derive (2.5).) Another closely related formula is representation (5.5) from [40]:

$$\varphi(z) = z^2 \int_0^\infty e^{izx} \chi(x) dx, \quad e^{-\eta x}\chi(x) \in L_{p \times p}^2(0, \infty) \text{ for all } \eta > 0, \tag{2.14}$$

where

$$\chi(x) := -2i \int_0^x s(t)^* dt. \tag{2.15}$$

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According to p. 341 in [40] the operators S_l admit a triangular factorisation

$$S_l = E_l^{-1}(E_l^{-1})^*, \quad E_l = I + \int_0^x E(x,t) \cdot dt, \tag{2.16}$$

$$E_l^{-1} = I + \int_0^x \Gamma(x,t) \cdot dt, \quad \sup_{0<t<x<l}(\|E(x,t)\| + \|\Gamma(x,t)\|) < \infty, \tag{2.17}$$

$$E_l^{-1}\theta_1(x/2) = \frac{1}{\sqrt{2}}[2s(x) \quad I_p] \quad (0 < x < l), \tag{2.18}$$

where $E_l, E_l^{-1} \in \{L_p^2(0,l), L_p^2(0,l)\}$ and E_l^{-1} is applied to $\theta_1(x/2)$ columnwise. (Note that the expression for the triangular factor V of S_l^{-1} on p. 341 in [40] is slightly more complicated but according to [44] we can put $\kappa(x) \equiv 1/\sqrt{2}$ in that expression. Taking into account also that S_l is here two times less than S_l in [40] and so E_l is here $\sqrt{2}$ times larger than its analog V in [40], we get precisely (2.16).) By (2.16) and (2.17) we have

$$S_l^{-1} = E_l^* E_l, \quad l > 0, \tag{2.19}$$

and the matrix function E is locally bounded, that is, we can use equality (2.19) to determine pointwise the kernel of the integral operator in the left hand-side. Specifically, we get

$$\begin{aligned}(S_l^{-1}f)(x) =& f(x) + \int_0^x \left(E(x,t) + \int_x^l E(r,x)^* E(r,t)dr\right)f(t)dt \\ &+ \int_x^l \left(E(t,x)^* + \int_t^l E(r,x)^* E(r,t)dr\right)f(t)dt.\end{aligned} \tag{2.20}$$

Now, the next result follows from Theorem 2.2 and formula (2.20).

Theorem 2.4 *Let φ be the Weyl function of the matrix Dirac system (2.1), where the potential v satisfies (2.12). Then v is uniquely recovered from φ by the formula*

$$v\left(\frac{l}{2}\right) = 2i(S_l^{-1}k)(l), \tag{2.21}$$

where k and S_l are given by (2.5) and (2.6).

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P r o o f. By (2.19) we get $S_r^{-1} = E_r^* E_r$. Note also that $(E_r f)(t) = (E_l f)(t)$ for $0 \leq t \leq r \leq l$. Hence, formulas (2.8) and (2.18) imply

$$\theta_2'(x) = -\sqrt{2}\Big(E_l k\Big)(2x)^* \Big(E_l[2s \quad I_p]\Big)(2x) = -2\Big(E_l k\Big)(2x)^* \theta_1(x), \quad 2x < l. \tag{2.22}$$

In view of (2.10) and the second equality in (2.11) one can see that

$$\theta_2' J \theta_1^* = -\theta_2 J(\theta_1')^* = -iv^*. \tag{2.23}$$

By the first equality in (2.7) and the second equality in (2.9) we have

$$\theta_1 J \theta_1^* \equiv I_p. \tag{2.24}$$

From (2.22)-(2.24) we obtain $v(x) = 2i\Big(E_l k\Big)(2x)$ or, equivalently,

$$v\Big(\frac{x}{2}\Big) = 2i\Big(E_l k\Big)(x), \quad 0 < x < l. \tag{2.25}$$

In other words, for almost all values of x we have

$$v\Big(\frac{x}{2}\Big) = 2i\Big(k(x) + \int_0^x E(x,t)k(t)dt\Big). \tag{2.26}$$

Putting $x = l$ in (2.20) and (2.26) we see that (2.21) is true. ∎

Formulas related to (2.21) can be found in [2, 16] for the cases of continuous v and of a skew-self-adjoint Dirac system, respectively.

The case of a continuous v was treated in [2]. If v is continuous, then by Theorem 1.2 in [2], the matrix function k is also continuous with a possible jump discontinuity at the origin. Moreover, it follows from the proof of Theorem 1.2 in [2] that the kernels $E(x,t)$ and $\Gamma(x,t)$ of the operators E_l and E_l^{-1}, respectively, are continuous in the triangles $0 \leq t \leq x \leq l$. In particular, $E(x,0)$ is well-defined. According to formulas (1.13) and (2.12) from [2] we have

$$v(x) = -2iE(2x,0). \tag{2.27}$$

Moreover, since $E(x,t)$ is continuous, formula (2.25) holds pointwise. By applying E_l^{-1} to both sides of (2.25) we get the following analog of the representation of A-amplitude of Schrödinger operator in [5]. (See formula (3.11) in [5], which is essential in the proof of Theorem 1 in the same paper.)

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Corollary 2.5 *Let the potential v of system (2.1) be continuous. Then the matrix function k given by (2.5) and (2.6) admits the representation*

$$k(2x) = -\frac{i}{2}\Big(v(x) + 2\int_0^x \Gamma\big(2x, 2t\big)v(t)dt\Big)$$

$$= -\frac{i}{2}\Big(v(x) + 2\int_0^x \Gamma\big(2x, 2(x-t)\big)v(x-t)dt\Big). \qquad (2.28)$$

The matrix functions k and Γ are continuous and (2.28) holds for all $x \in [0, \infty)$.

3 Canonical system. Explicit formulas

3.1 Canonical system and Schur coefficients

In this section we consider a canonical system

$$\frac{d}{dx}w(x, z) = izJH(x)w(x, z), \quad H(x) = \beta(x)^*\beta(x), \quad x \geq 0, \qquad (3.1)$$

where J is given by (2.9), β is an $p \times m$ ($m = 2p$) matrix function such that

$$\beta(x)J\beta(x)^* = D = \text{diag}\{d_1, d_2, \ldots, d_p\}, \qquad (3.2)$$

and diag denotes a diagonal matrix. Weyl function of the canonical system on the semi-axis $x \geq 0$ is a $p \times p$ holomorphic matrix function $\varphi(z)$, which satisfies the condition [49]

$$\int_0^\infty \big[\ I_p \quad i\varphi(z)^*\ \big]\, w(x, z)^*H(x)w(x, z)\begin{bmatrix} I_p \\ -i\varphi(z) \end{bmatrix}dx < \infty, \quad z \in \mathbb{C}_+,$$
$$(3.3)$$

where w is the fundamental solution of (3.1) normalized by $w(0, z) = I_{2p}$. The Dirac system of the previous section, is equivalent [40] to canonical system

$$\frac{d}{dx}\widehat{w}(x, z) = izJ\widehat{H}(x)\widehat{w}(x, z), \quad \widehat{H}(x) = \widehat{u}(x, 0)^*\widehat{u}(x, 0), \quad x \geq 0. \qquad (3.4)$$

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Taking into account (2.9) one can see that a simple multiplication $w(x,z) = e^{izx}\widehat{w}(x,z)$ of solution $\widehat{w}$ by scalar factor transforms system (3.4) into system (3.1), where

$$D = 2I_p, \quad \beta(x) = \sqrt{2}\theta_1(x), \tag{3.5}$$

and θ_1 is given by (2.7). Nevertheless, additional Weyl functions appear as a result of this transformation and the Weyl theory of system (3.1) requires some separate study even in this particular case.

Before we turn to the general case (3.2), we consider the canonical system (3.1), (3.5) generated by Dirac system in greater detail. Separating β into two $p \times p$ blocks, namely β_1 and β_2, and using (3.2) and (3.5) we obtain

$$\det \beta_2(x) \neq 0, \quad \Re\big(\beta_2(x)^{-1}\beta_1(x)\big) > 0. \tag{3.6}$$

By (2.1), (2.7), and (2.9) we get

$$\beta'(x)J\beta(x)^* = 2\theta_1'(x)J\theta_1(x)^* = 2iv(x)[0 \quad I_p]j\begin{bmatrix} I_p \\ 0 \end{bmatrix} = 0. \tag{3.7}$$

According to (3.7) we have $\beta_1'\beta_2^* + \beta_2'\beta_1^* = 0$, that is,

$$\beta_1'(x) = -\beta_2'(x)\beta_1(x)^*\big(\beta_2(x)^*\big)^{-1}. \tag{3.8}$$

It follows from (3.8) that $\big(\beta_2^{-1}\beta_1\big)' = -\beta_2^{-1}\beta_2'\big(\beta_2^{-1}\beta_1 + \beta_1^*(\beta_2^*)^{-1}\big)$. Moreover, formulas (2.4), (2.7), and (3.5) imply $\beta_2(0) = I_p$. Hence, we can recover β_2 from $\beta_2^{-1}\beta_1$ using the following differential equation and initial condition

$$\beta_2' = -\beta_2\big(\beta_2^{-1}\beta_1\big)'\big(\beta_2^{-1}\beta_1 + \beta_1^*(\beta_2^*)^{-1}\big)^{-1}, \quad \beta_2(0) = I_p. \tag{3.9}$$

Remark 3.1 *Note that $\beta_2^{-1}\beta_1$ is the continuous analog of the (well-known in the discrete case) Schur coefficients. Similar to the case of Toeplitz matrices, the function $\beta_2^{-1}\beta_1$ has the property $\Re\big(\beta_2^{-1}\beta_1\big) > 0$. Moreover, system (3.1) corresponding to operators with difference kernels is uniquely recovered from $\beta_2^{-1}\beta_1$ via formulas (3.9) and $\beta_1 = \beta_2\big(\beta_2^{-1}\beta_1\big)$.*

3.2 Explicit formulas

In the case of the rational Weyl functions and corresponding Hamiltonians H one can obtain explicit formulas for solutions of the direct and inverse problems using a GBDT version of the Bäcklund-Darboux transformation [38,39]. See [8,13,19,22,24,31,38,52] and references therein for various versions of the Bäcklund-Darboux transformation and commutation methods. For explicit formulas via inversion of semiseparable operators see, for instance, [2,16,43]. Here we shall apply the procedure from [38] to the initial system

$$\frac{d}{dx}w_0(x,z) = izJH_0w_0(x,z), \quad H_0 \equiv \begin{bmatrix} D/2 \\ I_p \end{bmatrix} \begin{bmatrix} D/2 & I_p \end{bmatrix} \quad (\det D \neq 0), \tag{3.10}$$

where $D = D^* = \mathrm{diag}\{d_1, d_2, \ldots, d_p\}$ is a fixed diagonal matrix. (Notice that J in (3.1) and in (3.10) is slightly different from J in [38].)

We consider a fixed integer $n > 0$, an $n \times n$ matrix α and two $n \times p$ matrices $\Lambda_1(0)$ and $\Lambda_2(0)$, such that

$$\alpha - \alpha^* = i\Lambda(0)J\Lambda(0)^*, \quad \Lambda(0) := [\Lambda_1(0) \quad \Lambda_2(0)]. \tag{3.11}$$

We now introduce matrix functions $\Lambda(x)$ and $\Sigma(x)$ with the equalities

$$\frac{d}{dx}\Lambda = -i\alpha\Lambda JH_0, \quad \Lambda(0) = [\Lambda_1(0) \quad \Lambda_2(0)]; \tag{3.12}$$

$$\frac{d}{dx}\Sigma = \Psi_1\Psi_1^*, \quad \Sigma(0) = I_n, \quad \Psi_k(x) := \Lambda(x)\begin{bmatrix} I_p \\ (-1)^{k+1}D/2 \end{bmatrix}, \quad k = 1, 2. \tag{3.13}$$

The first two equalities in (3.13) imply $\Sigma(x) \geq I_n$. Thus, Σ is invertible and we can put

$$H(x) = v_0(x)^*H_0v_0(x), \tag{3.14}$$

where

$$\frac{d}{dx}v_0 = -q_0v_0, \quad v_0(0)Jv_0(0)^* = J, \tag{3.15}$$

$$q_0(x) := J\Lambda(x)^*\Sigma(x)^{-1}\Lambda(x)JH_0 - JH_0J\Lambda(x)^*\Sigma(x)^{-1}\Lambda(x). \tag{3.16}$$

The transfer matrix function, which we use in GBDT, has the form

$$w_\alpha(x,z) = I_m - iJ\Lambda(x)^*\Sigma(x)^{-1}(\alpha - zI_n)^{-1}\Lambda(x). \tag{3.17}$$

In view of the second equality in (3.10), the first equality in (3.12) and the third equality in (3.13), we see that $\left(\frac{d}{dx}\Lambda\right)J\Lambda^* = -i\alpha\Psi_1\Psi_1^*$. Therefore, taking into account (3.11) and (3.13) we get

$$\frac{d}{dx}\left(\alpha\Sigma - \Sigma\alpha^*\right) = i\frac{d}{dx}\left(\Lambda J\Lambda^*\right), \quad \alpha\Sigma(0) - \Sigma(0)\alpha^* = i\Lambda(0)J\Lambda(0)^*.$$

The two equalities above imply

$$\alpha\Sigma(x) - \Sigma(x)\alpha^* = i\Lambda(x)J\Lambda(x)^*. \tag{3.18}$$

By direct calculation [45, 49], it follows from (3.17) and (3.18) that

$$w_\alpha(x, \overline{z})^* J w_\alpha(x, z) = J. \tag{3.19}$$

It follows from (3.1) and (3.15), respectively, that

$$w(x, \overline{z})^* J w(x, z) = J, \quad v(x)^* J v(x) = J. \tag{3.20}$$

According to Theorem 1 in [38] the fundamental solution w of the canonical system given by (3.1) and (3.14)-(3.16) is expressed via w_α:

$$w(x, z) = v_0(x)^{-1} w_\alpha(x, z) w_0(x, z) w_\alpha(0, z)^{-1} v_0(0), \tag{3.21}$$

where w_0 is the fundamental solution of the initial system (3.10). Here the fundamental solutions are normalized by the initial condition $w(0, z) = w_0(0, z) = I_m \ (m = 2p)$.

Remark 3.2 *If $\det\alpha \neq 0$, by formula (8) in [38] and by formula (3.15) we have*

$$v_0(x) = w_\alpha(x, 0)M, \quad M := v_0(0), \quad MJM^* = J. \tag{3.22}$$

Moreover, for the case of the initial system (3.10) the matrix functions w_0, Λ and Σ can be constructed explicitly. Hence, assuming $\det\alpha \neq 0$, one gets explicit expressions for $H(x)$ and $w(x, z)$ via formulas (3.14), (3.17), (3.21), and (3.22).

Indeed, put

$$Z = \begin{bmatrix} I_p & I_p \\ D/2 & -D/2 \end{bmatrix}, \quad \breve{D} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.23}$$

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It follows that

$$Z^{-1} = \mathrm{diag}\{D^{-1},\, D^{-1}\} \begin{bmatrix} D/2 & I_p \\ D/2 & -I_p \end{bmatrix}. \tag{3.24}$$

In view of (3.10), (3.23), (3.24) and normalization condition $w_0(0,z) = I_m$, it is immediately apparent that

$$w_0(x,z) = Z e^{izx\breve{D}} Z^{-1}. \tag{3.25}$$

According to (3.12), to the second equality in (3.10) and to the third equality in (3.13), we have

$$\frac{d}{dx}\Psi_1 = -i\alpha\Psi_1 D, \quad \Psi_1(0) = \Lambda_1(0) + \frac{1}{2}\Lambda_2(0)D, \quad \Psi_2(x) \equiv \Lambda_1(0) - \frac{1}{2}\Lambda_2(0)D. \tag{3.26}$$

The expression for Ψ_1 follows directly from (3.26):

$$\Psi_1(x) = [\exp(-id_1 x\alpha)f_1 \quad \exp(-id_2 x\alpha)f_2 \quad \ldots \quad \exp(-id_p x\alpha)f_p], \tag{3.27}$$

$$[f_1 \quad f_2 \quad \ldots \quad f_p] := \Psi_1(0). \tag{3.28}$$

Formulas (3.26) and (3.27) provide explicit expressions for Ψ_1 and Ψ_2. Finally, from (3.13) and (3.23) we obtain

$$\Lambda(x) = [\Psi_1(x) \quad \Psi_2(x)]Z^{-1}, \quad \Sigma(x) = I_n + \int_0^x \Psi_1(t)\Psi_1(t)^* dt. \tag{3.29}$$

Definition 3.3 *A canonical system given by (3.1) and (3.14), where H_0 is defined in (3.10), v_0 is defined by (3.15) and (3.16), and $\Lambda(x)$ and $\Sigma(x)$ in (3.16) are defined by (3.12) and (3.13) or, equivalently, by (3.26)-(3.29), is said to be determined by the parameter matrices α and $\Lambda(0)$ such that (3.11) holds.*

Further we always assume that (3.11) is valid.

3.3 Direct problem: explicit solutions

Remark 3.2 leads us to our next proposition.

Proposition 3.4 *Let $v_0(0) = I_m$ $(m = 2p)$. Then the matrix functions*

$$\varphi(z) = -\frac{i}{2}D + \Psi_1(0)^*(\gamma - zI_n)^{-1}\Psi_2, \tag{3.30}$$

$$\widehat{\varphi}(z) = \frac{i}{2}|D| + \widehat{\Psi}_1(0)^*(\widehat{\gamma} - zI_n)^{-1}\widehat{\Psi}_2, \tag{3.31}$$

where

$$\gamma =: \alpha - i\Psi_2\Lambda_2(0)^*, \quad \Psi_1(0) = \Lambda_1(0) + \frac{1}{2}\Lambda_2(0)D, \tag{3.32}$$

$$\Psi_2 = \Lambda_1(0) - \frac{1}{2}\Lambda_2(0)D \equiv \Lambda_1(x) - \frac{1}{2}\Lambda_2(x)D, \tag{3.33}$$

$$\widehat{\gamma} =: \alpha - i\widehat{\Psi}_2\Lambda_2(0)^*, \quad \widehat{\Psi}_1(0) = \Lambda_1(0) - \frac{1}{2}\Lambda_2(0)|D|, \tag{3.34}$$

$$\widehat{\Psi}_2 = \Lambda_1(0) + \frac{1}{2}\Lambda_2(0)|D|, \quad |D| = \mathrm{diag}\{|d_1|, |d_2|, \ldots, |d_p|\} \tag{3.35}$$

are Weyl functions of the canonical system determined by α and $\Lambda(0)$.
  If $D < 0$, then the Weyl functions $\varphi(z)$ and $\widehat{\varphi}(z)$ coincide and, moreover, the Weyl function is unique.
  The matrices γ and $\widehat{\gamma}$ given by (3.32) and (3.34), respectively, satisfy matrix identities

$$\gamma - \gamma^* = i\Lambda_2(0)D\Lambda_2(0)^*, \quad \widehat{\gamma}^* - \widehat{\gamma} = i\big(\widehat{\Psi}_2 - \widehat{\Psi}_1(0)\big)|D|^{-1}\big(\widehat{\Psi}_2 - \widehat{\Psi}_1(0)\big)^*, \tag{3.36}$$

that is, for $D < 0$ we obtain $\sigma(\gamma) \cap \mathbb{C}_+ = \emptyset$ and φ given by (3.30) does not have singularities in $\mathbb{C}_+$.

P r o o f. The proof that φ is a Weyl function is somewhat similar to the proof of the corresponding fact for the self-adjoint Dirac system [21]. Put

$$\Omega(z) := w_\alpha(0, z)Z, \quad \Omega_2(z) = \begin{bmatrix} \omega_1(z) \\ \omega_2(z) \end{bmatrix} := \Omega(z)\begin{bmatrix} 0 \\ I_p \end{bmatrix}, \tag{3.37}$$

where ω_k are $p \times p$ blocks of Ω_2. In view of (3.21), (3.25) and (3.37) we have

$$w(x, z)\Omega_2(z) = v_0(x)^{-1}w_\alpha(x, z)Z\begin{bmatrix} 0 \\ I_p \end{bmatrix} = v_0(x)^{-1}w_\alpha(x, z)\begin{bmatrix} I_p \\ -D/2 \end{bmatrix}. \tag{3.38}$$

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Notice also that by (3.13) and (3.17) the equality

$$Q(x) := \begin{bmatrix} D/2 & I_p \end{bmatrix} w_\alpha(x, z) \begin{bmatrix} I_p \\ -D/2 \end{bmatrix} = -i\Psi_1(x)^*\Sigma(x)^{-1}(\alpha - zI_n)^{-1}\Psi_2 \tag{3.39}$$

holds. From (3.10), (3.14), (3.38), and (3.39) it follows that

$$\Omega_2(z)^*w(x, z)^*H(x)w(x, z)\Omega_2(z) = Q(x)^*Q(x) \tag{3.40}$$
$$= \Psi_2^*(\alpha^* - \overline{z}I_n)^{-1}\Sigma(x)^{-1}\Psi_1(x)\Psi_1(x)^*\Sigma(x)^{-1}(\alpha - zI_n)^{-1}\Psi_2.$$

According to (3.13) we have

$$\frac{d}{dx}\Sigma^{-1} = -\Sigma^{-1}\Psi_1\Psi_1^*\Sigma^{-1}, \quad \Sigma(0) = I_n, \quad \Sigma(x) \geq I_n. \tag{3.41}$$

Hence, from (3.40) it follows that

$$\int_0^r \Omega_2(z)^*w(x, z)^*H(x)w(x, z)\Omega_2(z)dx = \Psi_2^*(\alpha^* - \overline{z}I_n)^{-1} \tag{3.42}$$
$$\times \big(\Sigma(0)^{-1} - \Sigma(r)^{-1}\big)(\alpha - zI_n)^{-1}\Psi_2 \leq \Psi_2^*(\alpha^* - \overline{z}I_n)^{-1}(\alpha - zI_n)^{-1}\Psi_2.$$

Compare (3.3) and (3.42) to see that the function

$$\varphi(z) = i\omega_2(z)\omega_1(z)^{-1}, \tag{3.43}$$

where ω_k are the blocks of Ω_2, satisfies (3.3) (excluding, possibly, a finite number of points), i.e., φ of the form (3.43) is a Weyl function of system (3.1), (3.14).

Next, let us show that the right-hand sides of (3.30) and (3.43) coincide. Since $v_0(0) = I_m$ and $\Sigma(0) = I_n$, we use (3.17), (3.23), and the third equality in (3.13) to rewrite (3.37) in the form

$$\omega_1(z) = I_p - i\Lambda_2(0)^*(\alpha - zI_n)^{-1}\Psi_2, \quad \omega_2(z) = -\frac{1}{2}D - i\Lambda_1(0)^*(\alpha - zI_n)^{-1}\Psi_2. \tag{3.44}$$

The following procedure is a standard one in system theory. Rewrite the first equality in (3.32) as

$$i\Psi_2\Lambda_2(0)^* = \alpha - \gamma = (\alpha - zI_p) - (\gamma - zI_p) \tag{3.45}$$

to check that

$$(I_p - i\Lambda_2(0)^*(\alpha - zI_n)^{-1}\Psi_2)(I_p + i\Lambda_2(0)^*(\gamma - zI_n)^{-1}\Psi_2) = I_p.$$

In other words, we have

$$\omega_1(z)^{-1} = I_p + i\Lambda_2(0)^*(\gamma - zI_n)^{-1}\Psi_2. \qquad (3.46)$$

From (3.44)-(3.46) it follows that

$$i\omega_2(z)\omega_1(z)^{-1} = -\frac{i}{2}D + \Psi_1(0)^*(\gamma - zI_p)^{-1}\Psi_2, \qquad (3.47)$$

that is, the function given by (3.30) is a Weyl function.

Now, consider the matrix function $\widehat{\varphi}$, given by (3.31). If $D < 0$, we have $\varphi = \widehat{\varphi}$ and the case of φ is as above. If the inequality $D < 0$ does not hold, we can introduce two diagonal matrices $P_j = P_j^* = P_j^2$ $(j = 1, 2)$ such that

$$P_1 + P_2 = I_p, \quad D(P_1 - P_2) = |D|. \qquad (3.48)$$

Here P_1 has nonzero entry (entry equal to 1), when the entry of D in the same row and column is positive, and P_2 has nonzero entry (entry equal to 1), when the corresponding entry of D is negative. Let us show that for $\Im z \geq \|\alpha\|$ we have

$$\int_0^\infty [P_1 \quad P_2]\Omega(z)^*w(x,z)^*H(x)w(x,z)\Omega(z)\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} dx < \infty. \qquad (3.49)$$

It suffices to show that

$$\int_0^\infty [0 \quad P_2]\Omega(z)^*w(x,z)^*H(x)w(x,z)\Omega(z)\begin{bmatrix} 0 \\ P_2 \end{bmatrix} dx < \infty, \qquad (3.50)$$

which is immediately evident from (3.42), and that

$$\int_0^\infty [P_1 \quad 0]\Omega(z)^*w(x,z)^*H(x)w(x,z)\Omega(z)\begin{bmatrix} P_1 \\ 0 \end{bmatrix} dx < \infty. \qquad (3.51)$$

To prove (3.51) note that similar to (3.40) one obtains

$$[P_1 \quad 0]\Omega(z)^*w(x,z)^*H(x)w(x,z)\Omega(z)\begin{bmatrix} P_1 \\ 0 \end{bmatrix} = Q_1(x)^*Q_1(x), \qquad (3.52)$$

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$$Q_1(x) = \left(D - i\Psi_1(x)^*\Sigma(x)^{-1}(\alpha - zI_n)^{-1}\Psi_1(x)\right)e^{izxD}P_1. \tag{3.53}$$

It is an immediate consequence of the above that the entries of $De^{izxD}P_1$ belong $L^2(0,\infty)$. It follows from (3.27) that the entries of $\Psi_1(x)e^{izxD}P_1$ are bounded for $\Im z \geq \|\alpha\|$. Because of (3.41) the entries of $\Psi_1(x)^*\Sigma(x)^{-1}$ belong $L^2(0,\infty)$. Thus, in view of (3.53) the entries of Q_1 belong $L^2(0,\infty)$ ($\Im z \geq \|\alpha\|$), and using (3.52) we see that (3.51) is true. Hence (3.49) is also valid. Substitute $\widehat{\varphi}$ of the form

$$\widehat{\varphi}(z) = i\widehat{\omega}_2(z)\widehat{\omega}_1(z)^{-1}, \quad \begin{bmatrix} \widehat{\omega}_1(z) \\ \widehat{\omega}_2(z) \end{bmatrix} := \Omega(z)\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{3.54}$$

into the left-hand side of (3.3). By (3.49) the function $\widehat{\varphi}$ of the form (3.54) satisfies (3.3) in the domain $\Im z > \|\alpha\|$.

According to (3.37), (3.48), and (3.54) we have

$$\begin{bmatrix} \widehat{\omega}_1(z) \\ \widehat{\omega}_2(z) \end{bmatrix} = w_\alpha(0,z)\begin{bmatrix} I_p \\ |D|/2 \end{bmatrix}. \tag{3.55}$$

In view of (3.55), the formula

$$i\widehat{\omega}_2(z)\widehat{\omega}_1(z)^{-1} = \frac{i}{2}|D| + \widehat{\Psi}_1(0)^*(\widehat{\gamma} - zI_n)^{-1}\widehat{\Psi}_2 \tag{3.56}$$

can be proven quite similar to (3.47). Equation (3.31) is an immediate consequence of (3.54) and (3.56), and so $\widehat{\varphi}$ given by (3.31) satisfies (3.3).

To prove the uniqueness of the Weyl function for $D < 0$, notice that

$$H_0 + cJ = \begin{bmatrix} D/2 \\ 2D^{-1}\left(\frac{1}{2}D + cI_p\right) \end{bmatrix}\begin{bmatrix} D/2 & 2\left(\frac{1}{2}D + cI_p\right)D^{-1} \end{bmatrix}$$
$$- \begin{bmatrix} 0 & 0 \\ 0 & 4cD^{-1}\left(D + cI_p\right)D^{-1} \end{bmatrix}. \tag{3.57}$$

When $D < 0$ and the scalar $c > 0$ is sufficiently small, formula (3.57) implies $H_0 \geq -cJ$. Hence, because of (3.14)-(3.16) we have

$$H(x) \geq -cv_0(x)^*Jv_0(x) = -cJ. \tag{3.58}$$

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Using (3.1) we obtain

$$-c\frac{d}{dx}\big(w(x,z)^*Jw(x,z)\big) = ic(\overline{z}-z)w(x,z)^*H(x)w(x,z) \geq 0, \quad z \in \mathbb{C}_+.$$
(3.59)

From (3.58) and (3.59) it follows that

$$w(x,z)^*H(x)w(x,z) \geq -cw(x,z)^*Jw(x,z) \geq -cw(0,z)^*Jw(0,z) = -cJ.$$
(3.60)

Formula (3.60) implies that

$$\int_0^r \begin{bmatrix} I_p & -I_p \end{bmatrix} w(x,z)^*H(x)w(x,z) \begin{bmatrix} I_p \\ -I_p \end{bmatrix} dx \geq 2crI_p.$$
(3.61)

Denote by $L \subset \mathbb{C}^m$ $(m = 2p)$ the subspace of vectors f such that

$$\int_0^r f^*w(x,z)^*H(x)w(x,z)f\,dx < \infty.$$

Inequality (3.61) implies $\dim L \leq p$, which implies the uniqueness of the Weyl function.

Finally, formulas (3.11) and (3.32)-(3.35) imply (3.36). ∎

Remark 3.5 *Representations like (3.30) and (3.32) are called realizations in Control Theory.*

Remark 3.6 *It follows from the proof of Proposition 3.4 that φ satisfies (3.3) in $\mathbb{C}_+ \setminus \big(\sigma(\alpha) \cup \sigma(\gamma)\big)$ and $\widehat{\varphi}$ satisfies (3.3) in $\{z : \Im z > \|\alpha\|\}$.*

3.4 Inverse problem: explicit solution

The following theorem allows us to recover H explicitly from the rational Weyl functions.

Theorem 3.7 *Let the diagonal $p \times p$ matrix D be negative (i.e., let $D < 0$). If φ is a rational matrix function such that*

$$\lim_{z\to\infty} \varphi(z) = \frac{i}{2}|D|, \quad \Im\varphi(z) \geq 0 \quad (z \in \mathbb{C}_+),$$
(3.62)

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then φ admits a realization

$$\varphi(z) = \frac{i}{2}|D| + \Psi_1(0)^*(\gamma - zI_n)^{-1}\Psi_2, \tag{3.63}$$

where $\Psi_1(0)$ and Ψ_2 are $n \times p$ matricies, $n \in \mathbb{N}$, and $n \times n$ matrix γ satisfies the matrix identity

$$\gamma - \gamma^* = i(\Psi_1(0) - \Psi_2)D^{-1}(\Psi_1(0) - \Psi_2)^*. \tag{3.64}$$

Moreover φ is the Weyl function of the canonical system determined (in the sense of Definition 3.3) by the parameter matrices

$$\alpha = \gamma + i\Psi_2\Lambda_2(0)^*, \quad \Lambda_1(0) = \frac{1}{2}(\Psi_1(0) + \Psi_2), \quad \Lambda_2(0) = (\Psi_1(0) - \Psi_2)D^{-1}. \tag{3.65}$$

P r o o f. Realization (3.63) follows directly from Proposition 4.1 in [43] (see also Theorem 5.2 in [22]). The identity (3.11) follows from (3.64) and (3.65), that is, the requirement (3.11) for parameter matrices is fulfilled. Moreover, (3.65) implies relations (3.32) and (3.33). Now, compare (3.30) and (3.63) to see that φ is the Weyl function by Proposition 3.4. ■

4   Canonical system. General formulas

In the Krein paper [27] (see also [1,2]) Krein system was treated as a system generated by an accelerant $k(x) = k(-x)^*$, where k was the kernel of operator S (with difference kernel) of the form (2.6). In this section we assume that the diagonal matrix D is negative (i.e., $D < 0$). In a similar way to [27], operators with $|D|$-difference kernel generate other subclasses of canonical systems (see [43, 48, 49] and references therein):

Proposition 4.1 *Let $k(x) = \{k_{ij}(x)\}_{i,j=1}^p$ be a $p \times p$ matrix function such that*

$$k(x) \in L^2_{p\times p}(0,l), \quad k(-x) = k(x)^*, \quad S_l, S_l^{-1} \in \{L^2_p(0,l), L^2_p(0,l)\}, \tag{4.1}$$

$$S_l := I + \int_0^l \{k_{ij}(d_j t - d_i x)\}_{i,j=1}^p \cdot dt > 0, \quad \text{for all} \quad 0 < l < \infty, \tag{4.2}$$

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that is, operators S_l determined via k are positive, bounded, and boundedly invertible. Then operators S_l admit a triangular factorization

$$S_l^{-1} = E_l^* E_l, \quad E_l = I + \int_0^x E(x,t) \cdot dt \in \{L_p^2(0,\ l),\ L_p^2(0,\ l)\}, \qquad (4.3)$$

where $E(x,t)$ is a Hilbert-Schmidt kernel.

Moreover, k generates, in terms of S and E, the canonical system (3.1) such that (3.2) holds, and the Hamiltonian $H = \beta^*\beta$ of the canonical system is given via

$$\beta(x) = (E_l \Pi)(x), \quad \Pi(x) := \left[D\{s_{ij}(|d_i|x)\}_{i,j=1}^p \qquad I_p \right], \qquad (4.4)$$

$$s(x) := \frac{1}{2} I_p + |D|^{-1} \int_0^x k(t)dt. \qquad (4.5)$$

This $H(x)$ is summable on all finite intervals $(0,\ l)$.

The fundamental solution of the canonical system is given by

$$w(l,z) = I_{2p} + izJ\Pi_l^* S_l^{-1}(I - zA)^{-1}\Pi_l, \quad A, \Pi_l \in \{\mathbb{C}^{2p},\ L_p^2(0,\ l)\}; \qquad (4.6)$$

$$A = A_l = iD \int_0^x \cdot dt; \quad \Pi_l g = \Pi(x)g, \quad g \in \mathbb{C}^{2p}, \qquad (4.7)$$

where $\mathbb{C}^{2p}$ is the $2p$-dimensional vector space.

P r o o f. According to [46] (Ch. 6) the operators A, S_l, Π_l given by (4.2) and (4.7) satisfy the operator identity

$$AS_l - S_l A^* = i\Pi_l J\Pi_l^*, \qquad (4.8)$$

that is, they form an S-node [45, 46, 48, 49]. By Theorem 4.2.1 [49] there is a system corresponding to the family A, S_l, Π_l ($0 < l < \infty$) of S-nodes, and w given by (4.6) is its fundamental solution. Moreover, if $B(l) := \Pi_l^* S_l^{-1}\Pi_l$ is differentiable, this system is canonical, and its Hamiltonian is given by

$$H(l) = \frac{d}{dl}B(l) = \frac{d}{dl}\Pi_l^* S_l^{-1}\Pi_l. \qquad (4.9)$$

The factorisation $S_l^{-1} = (I + E_+)(I + E_-)$, where E_+ (E_-) is an upper (lower) triangular integral operator, is clear from the factorization result on p. 184

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in [23]. Now, taking into account $S_l = S_l^*$ it is easy to derive (4.3) (see, for instance, Section 4 in [43]). According to (4.3) and (4.9) we get $H = \beta^*\beta$, where β is given by (4.4). Note that $E(x, t)$ does not depend on l since the lower-upper factorisation of S_l is unique, and the lower-upper factorisation of S_l is unique because the lower-upper factorisation of I is unique.

It remains to be shown that the equality (3.2) holds for almost all values of x. For that purpose we multiply (4.8) by E_l from the left and by E_l^* from the right. Then, taking into account (4.3) and (4.4) we get:

$$E_l A E_l^{-1} - \left(E_l A E_l^{-1}\right)^* = i(E_l \Pi)J(E_l \Pi)^* = i\beta(x)J \int_0^l \beta(t)^* \cdot dt. \quad (4.10)$$

As $E_l A E_l^{-1}$ is a lower triangular operator, formula (4.10) implies

$$E_l A E_l^{-1} = i\beta(x)J \int_0^x \beta(t)^* \cdot dt. \quad (4.11)$$

Denote the kernel of E_l^{-1} by $\widehat{E}$ and rewrite the equality (4.11) in terms of the kernels of integral operators:

$$D + \int_t^x \left(D\widehat{E}(y, t) + E(x, y)D\right)dy + \int_t^x E(x, y)D \int_t^y \widehat{E}(r, t)drdy \quad (4.12)$$
$$= \beta(x)J\beta(t)^* \quad (t \leq x).$$

Equality (3.2) follows from (4.12). ∎

Definition 4.2 *The class of Hamiltonians $H(x)$ $(0 < x < \infty)$ of canonical systems generated by matrix functions k, which satisfy the conditions of Proposition 4.1, is denoted by $\mathcal{H}(D)$.*

We next consider some fixed $H \in \mathcal{H}(D)$ and turn our focus towards Möbius (also called linear-fractional) transformations

$$\varphi(z, l) = i\left(\mathcal{W}_{11}(z)\mathcal{P}_1(z) + \mathcal{W}_{12}(z)\mathcal{P}_2(z)\right)\left(\mathcal{W}_{21}(z)\mathcal{P}_1(z) + \mathcal{W}_{22}(z)\mathcal{P}_2(z)\right)^{-1}, \quad (4.13)$$

where

$$\mathcal{W}(l, z) = \{\mathcal{W}_{ij}(z)\}_{i,j=1}^2 = w(l, \overline{z})^*, \quad (4.14)$$

and the pairs $\{\mathcal{P}_1(z), \mathcal{P}_2(z)\}$ are pairs of $p \times p$ matrix functions, which are meromorphic in $\mathbb{C}_+$ and satisfy (excluding, possibly, a discrete set of points) the following relations

$$\mathcal{P}_1^* \mathcal{P}_1 + \mathcal{P}_2^* \mathcal{P}_2 > 0, \quad \mathcal{P}_1^* \mathcal{P}_2 + \mathcal{P}_2^* \mathcal{P}_1 \geq 0, \tag{4.15}$$

that is, $\{\mathcal{P}_1, \mathcal{P}_2\}$ are pairs of nonsingular matrix functions with property-J. The set of Möbius transformations (4.13), where the pairs $\{\mathcal{P}_1, \mathcal{P}_2\}$ vary and l is fixed, is denoted by $\mathcal{N}_l$. It follows from Statement 3 in [36] and interpolation results in [45] (see also their formulation in Theorem 1 from [36] or Chapter 1 in [47]) that there is a unique matrix function

$$\varphi = \bigcap_{l>0} \mathcal{N}_l, \tag{4.16}$$

and this φ admits representation

$$\varphi(z) = zD^2 \int_0^\infty e^{-izxD} \{s_{ij}(|d_i|x)\}_{i,j=1}^p dx \quad (z \in \mathbb{C}_+), \tag{4.17}$$

where s is given by (4.5). Moreover, according to Theorem 1 from [36] the scalar products $\langle S_l f, f \rangle$ $(0 < l < \infty)$ admit representation

$$\langle S_l f, f \rangle = \int_{-\infty}^\infty \left(\int_0^l e^{-itxD} f(x) dx \right)^* d\tau(t) \left(\int_0^l e^{-ityD} f(y) dy \right). \tag{4.18}$$

Here τ is the $p \times p$ non-decreasing matrix function from Herglotz representation of φ:

$$\varphi(z) = \mu z + \nu + \int_{-\infty}^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\tau(t) = \mu z + \nu + \int_{-\infty}^\infty \frac{1+tz}{t-z} \frac{d\tau(t)}{1+t^2}, \tag{4.19}$$

where $\mu \geq 0$ $(\mu = 0$ in our special case), $\nu = \nu^*$, and

$$\int_{-\infty}^\infty (1+t^2)^{-1} d\tau(t) < \infty. \tag{4.20}$$

Note that according to (3.1) and (4.14) we have

$$\frac{d}{dx} \left(w(x, \bar{z})^* J w(x, z) \right) = 0, \quad \text{and so} \quad \mathcal{W}(l, z) = J w(l, z)^{-1} J. \tag{4.21}$$

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From (4.21) we get

$$\left(\mathcal{W}(l,z)^*\right)^{-1} J \mathcal{W}(l,z)^{-1} = Jw(l,z)^* Jw(l,z) J. \qquad (4.22)$$

Moreover, taking into account (4.6) and (4.8) we obtain

$$w(l,z)^* Jw(l,z) = J + i(z - \overline{z})\Pi_l^*(I - \overline{z}A^*)^{-1} S_l^{-1}(I - zA)^{-1}\Pi_l. \qquad (4.23)$$

Using (4.4) and (4.7), separate Π_l into two blocks $\Phi_k \in \{\mathbb{C}^p, L_p^2(0, l)\}$:

$$\Phi_1 h = D\{s_{ij}(|d_i|x)\}_{i,j=1}^p h, \quad \Phi_2 h \equiv h \quad (h \in \mathbb{C}^p); \quad \Pi_l = [\Phi_1 \quad \Phi_2]. \qquad (4.24)$$

We omit index "l" in A and Φ_k as the choice of "l" is clear from the context. Finally, formulas (4.22)-(4.24) imply

$$r(l,z) := -[I_p \quad 0]\left(\mathcal{W}(l,z)^*\right)^{-1} J \mathcal{W}(l,z)^{-1} \begin{bmatrix} I_p \\ 0 \end{bmatrix} \qquad (4.25)$$

$$= i(\overline{z} - z)\Phi_2^*(I - \overline{z}A^*)^{-1} S_l^{-1}(I - zA)^{-1}\Phi_2 > 0 \quad (z \in \mathbb{C}_+).$$

By (4.16) and (4.25) the conditions of Proposition 9.1.5 from [49] are fulfilled, and hence φ satisfies (3.3).

Theorem 4.3 *Let $H \in \mathcal{H}(D)$ $(D < 0)$. Then the unique Weyl function of the corresponding canonical system (3.1) admits the representation*

$$\varphi(z) = -zD \int_0^\infty e^{izx} s(x)dx \qquad (z \in \mathbb{C}_+), \qquad (4.26)$$

where s is given by (4.5).

P r o o f. Representation (4.26) is clear from (4.17).

It remains only to prove that φ satisfying (3.3) is unique. Here, it would be of interest to give a proof connected with S-nodes to compare with the proof of uniqueness from Proposition 3.4. For this purpose let us consider $\Phi_2^*(I - \overline{z}A^*)^{-1} S_l^{-1}(I - zA)^{-1}\Phi_2$.

First, fix some $a \in \mathbb{R}_+$, put $l = na$ $(n \in \mathbb{N})$, and consider the scalar product in (4.18). It is clear that

$$\langle S_{na}f, f\rangle \leq p\sum_{j=1}^p \langle S_{na}f_j e_j, f_j e_j\rangle, \quad f =: \{f_j\}_{j=1}^p, \quad e_j := \{\delta_{ij}\}_{i=1}^p. \qquad (4.27)$$

Formula (4.18) and the equality below

$$\int_0^{na} e^{-ityD}\big(f_j(y)\mathrm{e}_j\big)dy = \sum_{k=1}^{n} e^{-i(k-1)tad_j} \int_0^{a} e^{-ityd_j} f_{jk}(y)dy\,\mathrm{e}_j, \qquad (4.28)$$

$$f_{jk}(y) := f_j\big(y + (k-1)a\big) \in L^2(0,a),$$

imply that

$$\langle S_{na}f_j\mathrm{e}_j, f_j\mathrm{e}_j\rangle \le n \sum_{k=1}^{n} \int_{-\infty}^{\infty} \Big| \int_0^{a} e^{-ityd_j} f_{jk}(y)dy \Big|^2 \mathrm{e}_j^* d\tau(t)\mathrm{e}_j. \qquad (4.29)$$

It follows from (4.18) and (4.29) that

$$\langle S_{na}f_j\mathrm{e}_j, f_j\mathrm{e}_j\rangle \le n\|S_a\| \sum_{k=1}^{n} \|f_{jk}\|^2 = n\|S_a\|\|f_j\|^2. \qquad (4.30)$$

By (4.27) and (4.30) we have $\langle S_{na}f, f\rangle \le pn\|S_a\|\|f\|^2$, and so we get

$$\|S_{na}\| \le pn\|S_a\|, \quad \text{i.e.,} \quad S_{na}^{-1} \ge \big(pn\|S_a\|\big)^{-1} I. \qquad (4.31)$$

One can easily check that $(I - zA)^{-1}\Phi_2 = e^{izxD}$. Thus, using (4.31) we derive

$$\Phi_2^*(I - \overline{z}A^*)^{-1} S_{na}^{-1}(I - zA)^{-1}\Phi_2 \ge c(n)I_p, \quad c(n) \to +\infty\ (n \to \infty). \qquad (4.32)$$

Now, let φ and $\widetilde{\varphi}$, such that $\psi(z) := \varphi(z) - \widetilde{\varphi}(z) \ne 0$ for some $z \in \mathbb{C}_+$, satisfy (3.3). This implies that

$$\int_0^{\infty} \begin{bmatrix} 0 & i\psi(z)^* \end{bmatrix} w(x,z)^* H(x)w(x,z) \begin{bmatrix} 0 \\ -i\psi(z) \end{bmatrix} dx < \infty. \qquad (4.33)$$

According to (3.1) and similar to (3.59) we have

$$\frac{d}{dx}\Big(w(x,z)^* J w(x,z)\Big) = i(z - \overline{z})w(x,z)^* H(x)w(x,z).$$

Hence, we rewrite (4.33) as

$$\sup_{l<\infty} \left(i(\overline{z} - z)^{-1} \begin{bmatrix} 0 & i\psi(z)^* \end{bmatrix} w(l,z)^* J w(l,z) \begin{bmatrix} 0 \\ -i\psi(z) \end{bmatrix} \right) < \infty. \qquad (4.34)$$

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In view of (4.23) inequality (4.34) implies

$$\sup_{n \in \mathbb{N}} \left(\psi(z)^* \Phi_2^* (I - \overline{z}A^*)^{-1} S_{na}^{-1} (I - zA)^{-1} \Phi_2 \psi(z) \right) < \infty,$$

which contradicts (4.32). ∎

Theorem 4.3 yields a solution of the inverse problem directly via Weyl function.

Theorem 4.4 *Let φ be the Weyl function of the canonical system such that its Hamiltonian $H \in \mathcal{H}(D)$ $(D < 0)$. Then we have*

$$e^{-\eta x} s(x) \in \left(L^1_{p \times p}(0, \infty) \bigcap L^2_{p \times p}(0, \infty) \right) \tag{4.35}$$

for any $\eta > 0$, and the matrix function $k(x)$ $(x > 0)$ is recovered via Fourier transform

$$k(x) = \frac{1}{2\pi} \frac{d}{dx} \left(e^{\eta x} \text{l.i.m.}_{a \to \infty} \int_{-a}^{a} e^{-i\zeta x} \frac{\varphi(\zeta + i\eta)}{\zeta + i\eta} d\zeta \right). \tag{4.36}$$

The Hamiltonian H is recovered from k as in Proposition 4.1 or, equivalently, using (4.9).

P r o o f. To prove (4.35) recall the first inequality in (4.31), which holds for all $a \in \mathbb{R}_+$. It follows that

$$\|S_l\| \leq \|S_{\lceil l \rceil}\| \leq p(l+1)\|S_1\| \leq C_1(l+1), \quad C_1 \in \mathbb{R}_+. \tag{4.37}$$

We next represent the operator $\Phi_1 = \Phi_{1,l}$ given by (4.24) in the form

$$\Phi_1 = \Upsilon_l(z) + i\Phi_2\varphi(z) = (I - zA)(I - zA)^{-1}\Upsilon_l(z) + i\Phi_2\varphi(z), \tag{4.38}$$
$$\Upsilon_l(z) := \Phi_1 - i\Phi_2\varphi(z).$$

Now, we need inequality (22) from [36]:

$$\|(I - zA)^{-1}\Upsilon_l(z)\| \leq \|S_l\|^{1/2} \sqrt{\|(\varphi(z) - \varphi(z)^*)/(z - \overline{z})\|}, \quad z \in \mathbb{C}_+. \tag{4.39}$$

Note that we can fix in (4.38) and (4.39) any $z \in \mathbb{C}_+$. Taking into account relations $\|\Phi_2\| = \sqrt{l}$ and $\|A\| < C_2 l$, we derive from (4.37)-(4.39) that

$$\|\Phi_1\| \leq C_3(l+1)^{3/2}. \tag{4.40}$$

24

Relation (4.35) follows from (4.24) and (4.40).

Rewrite (4.26) as

$$|D|^{-1}\varphi(\zeta + i\eta)/(\zeta + i\eta) = \int_0^\infty e^{i\zeta x}e^{-\eta x}s(x)dx \qquad (\zeta \in \mathbb{R},\ \eta > 0). \quad (4.41)$$

In view of the Plancherel theorem formulas (4.35) and (4.41) imply

$$|D|^{-1}\varphi(\zeta + i\eta)/(\zeta + i\eta) = \text{l.i.m.}_{a\to\infty} \int_0^a e^{i\zeta x}e^{-\eta x}s(x)dx, \qquad (4.42)$$

$$e^{-\eta x}s(x) = \frac{1}{2\pi}\text{l.i.m.}_{a\to\infty} \int_{-a}^a e^{-i\zeta x}|D|^{-1}\varphi(\zeta + i\eta)/(\zeta + i\eta)d\zeta \qquad (4.43)$$

for all fixed $\eta > 0$. Note that (4.43) holds for $x > 0$, that is, l.i.m. in (4.43) is considered in $L^2_{p\times p}(0, \infty)$. Finally, equalities (4.5) and (4.43) yield (4.36). ∎

Using Theorems 4.3 and 4.4 we obtain our next Borg-Marchenko-type result.

Theorem 4.5 *Let φ and $\widetilde{\varphi}$ be the Weyl functions of the canonical systems with Hamiltonians H and $\widetilde{H}$, respectively, where $H, \widetilde{H} \in \mathcal{H}(D)$ $(D < 0)$. Let the equalities*

$$e^{-iza}\left(\varphi(z) - \widetilde{\varphi}(z)\right) = O(1) \quad (z \to \infty) \qquad (4.44)$$

hold for all $a < l$ on some ray $\Im z/\Re z = c$ in $\mathbb{C}_+$. Then we have

$$H(x) \equiv \widetilde{H}(x) \quad \text{for} \quad 0 < x < l/d, \quad d := \max_{1 \le k \le p} |d_k|. \qquad (4.45)$$

P r o o f. We mark functions corresponding to $\widetilde{\varphi}$ with a tilde, e.g., $\widetilde{H}, \widetilde{s}$. Consider the entire matrix function

$$\omega(z) := \int_0^a e^{iz(x-a)}\left(s(x) - \widetilde{s}(x)\right)dx. \qquad (4.46)$$

It is clear that $\|\omega(z)\|$ is bounded in the closed lower semi-plane and tends to zero on some ray $\Im z/\Re z = c_1$ there. To estimate ω on the ray $\Im z/\Re z = c$ in $\mathbb{C}_+$ we use (4.26) and write

$$\omega(z) = e^{-iza}(z|D|)^{-1}\left(\varphi(z) - \widetilde{\varphi}(z)\right) - \int_a^\infty e^{iz(x-a)}\left(s(x) - \widetilde{s}(x)\right)dx. \quad (4.47)$$

25

From (4.35), (4.44), and (4.47) one gets the boundedness of $\|\omega(z)\|$ on the ray $\Im z/\Re z = c$. Hence, according to the Phragmen-Lindelöf theorem $\|\omega(z)\|$ is bounded in $\mathbb{C}_+$, and thus also in $\mathbb{C}$. As $\|\omega(z)\|$ tends also to zero on some ray, we see that $\omega(z) \equiv 0$, that is, $s(x) \equiv \widetilde{s}(x)$ for $x < a$. Since this fact is true for all $a < l$ the equalities

$$s(x) \equiv \widetilde{s}(x), \quad k(x) \equiv \widetilde{k}(x) \quad (0 < x < l) \tag{4.48}$$

follow. Now, take into account (4.2), (4.4), (4.9), and (4.48) to obtain (4.45). ∎

5 Interpolation of the Weyl function

As can also be seen from the cases treated in Sections 2-4 Weyl functions contain important information about systems, and their interpolation is of particular interest. One of the possible approaches is interpolation by rational Weyl functions. Recall that an inverse problem for rational Weyl functions of a subclass of canonical systems was solved explicitly in Subsection 3.4. In this section we shall consider another approach, namely an approach in the spirit of [35]. Theorem 4 from [35] (after reformulation for the matrix case and for the upper semi-plane) has the following form.

Theorem 5.1 *Let matrix function F admit the representation*

$$F(z) = \int_0^\infty e^{izx} f(x)dx, \quad e^{-\eta x} f(x) \in L^1_{p \times p}(0, \infty) \text{ for all } \eta > 0. \tag{5.49}$$

Then for any $\varepsilon > 0$ and $\Im z > \frac{1}{2} + \varepsilon$ we have

$$F(z) = \sum_{n=0}^\infty c_n\left(z + \frac{i}{2} - i\varepsilon\right) \sum_{q=0}^n a_{nq} F(iq + i\varepsilon), \tag{5.50}$$

where

$$a_{nq} = \frac{(-1)^q (n+q)!}{(q!)^2 (n-q)!}, \quad c_n(\lambda) = \frac{(2n+1) \prod_{q=1}^n (q - \frac{1}{2} + i\lambda)}{\prod_{q=0}^n (q + \frac{1}{2} - i\lambda)}. \tag{5.51}$$

26

The series in (5.50) converges uniformly on any compact subset in the semi-plane $\Im z > \frac{1}{2} + \varepsilon$ and we get

$$\left\| F(z) - \sum_{n=0}^{N} c_n \left(z + \frac{i}{2} - i\varepsilon \right) \sum_{q=0}^{n} a_{nq} F(iq + i\varepsilon) \right\| = O\left(N^{\frac{1}{2} + \varepsilon - \Im z} \right), \quad N \to \infty.$$

$$(5.52)$$

Our next theorem is an analogue for the Dirac system case of the main Theorem 5 from [35].

Theorem 5.2 *Let φ be the Weyl function of system (2.1), where v satisfies (2.12). Then for any $\varepsilon > 0$ and $\Im z > \frac{1}{2} + \varepsilon$ the matrix function φ admits the representation*

$$\varphi(z) = -z^2 \sum_{n=0}^{\infty} c_n \left(z + \frac{i}{2} - i\varepsilon \right) \sum_{q=0}^{n} (q + \varepsilon)^{-2} a_{nq} \varphi(iq + i\varepsilon). \quad (5.53)$$

Moreover, for $\Im z > \frac{1}{2} + \varepsilon$ we get

$$\left\| \varphi(z) + z^2 \sum_{n=0}^{N} c_n \left(z + \frac{i}{2} - i\varepsilon \right) \sum_{q=0}^{n} (q + \varepsilon)^{-2} a_{nq} \varphi(iq + i\varepsilon) \right\|$$

$$= O\left(N^{\frac{1}{2} + \varepsilon - \Im z} \right), \quad N \to \infty. \quad (5.54)$$

P r o o f. The theorem's statement follows from formula (2.14) and Theorem 5.1. ∎

Remark 5.3 *One can use various transformations of φ to change the interpolation set $\{z : z = iq + i\varepsilon, \ q \in \mathbb{N}_0\}$. The simplest example is $\varphi(z + z_0)$, where $z_0 \in (\mathbb{C}_+ \cup \mathbb{R})$. Noticing that $\varphi(z)/z^2$ satisfies the conditions of Theorem 5.1, we see that $\varphi(z + z_0)/(z + z_0)^2$ (as a function of z) also satisfies these conditions. Hence, we obtain*

$$\varphi(z + z_0) = -(z + z_0)^2 \sum_{n=0}^{\infty} c_n \left(z + \frac{i}{2} - i\varepsilon \right)$$

$$\times \sum_{q=0}^{n} (q + \varepsilon - iz_0)^{-2} a_{nq} \varphi(z_0 + iq + i\varepsilon). \quad (5.55)$$

27

Clearly, representation (4.26) enables us to apply Theorem 5.1 to Weyl functions of canonical systems.

In fact, interpolation formulas of (5.53) type are true for the Weyl functions of the much wider class of Dirac systems as well as for other classical and non-classical systems because the corresponding Weyl functions are either bounded or belong to the Nevanlinna (also called Herglotz) class, that is, $\Im\varphi(z) \geq 0$.

Indeed, according to Proposition 4.2 in [40] there is a unique Weyl function φ associated with a Dirac system on $[0, \infty)$ with a locally summable potential v. Moreover, this φ belongs to the Nevanlinna class and therefore admits a Herglotz representation (4.19).

Proposition 5.4 *Suppose $\varphi(z)$ $(z \in \mathbb{C}_+)$ belongs to the Nevanlinna class. Then the matrix functions $(z + i\delta)^{-2}\varphi(z + i\delta)$ $(\delta > 0)$ satisfy the conditions of Theorem 5.1.*

P r o o f. Put $z + i\delta = \xi + i\eta$ $(\xi \in \mathbb{R}, \ \eta > \delta)$. Then, in view of (4.19) and (4.20) (and for some $C_1, C_2 > 0$) we get the inequality

$$\left(\int_{-\infty}^{\infty} |\xi + i\eta|^{-4} |h^* \varphi(\xi + i\eta)h|^2 d\xi \right)^{1/2}$$

$$\leq \left(\int_{-\infty}^{\infty} \frac{1}{\xi^2 + \eta^2} \left| \int_{-\infty}^{\infty} \frac{\xi d\tau_h(t)}{(t - \xi - i\eta)(1 + t^2)} \right|^2 d\xi \right)^{1/2} + C_1$$

$$\leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left((t - \xi)^2 + \eta^2 \right)^{-\frac{1}{2}} \left((\zeta - \xi)^2 + \eta^2 \right)^{-\frac{1}{2}} d\xi \frac{d\tau_h(\zeta)}{1 + \zeta^2} \frac{d\tau_h(t)}{1 + t^2} \right)^{\frac{1}{2}}$$

$$+ C_1 \leq \int_{-\infty}^{\infty} \frac{d\tau_h(t)}{1 + t^2} \left(\int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + \delta^2} \right)^{1/2} + C_1 \leq C_2, \tag{5.56}$$

where $h \in \mathbb{C}^p$, $h^*h \leq 1$, $\tau_h(t) = h^*\tau(t)h$, and C_2 does not depend on h and on $\eta > \delta$. Therefore, the norms of entries of $(\xi + i\eta)^{-2}\varphi(\xi + i\eta)$ in $L^2(-\infty, \infty)$ are uniformly bounded for $\eta > \delta$. Thus, we can apply Theorem V from [33] and derive

$$(z + i\delta)^{-2}\varphi(z + i\delta) = \text{l.i.m.}_{a \to \infty} \int_0^a e^{izx} f(x) dx \quad (\Im z > 0), \tag{5.57}$$

28

where $f \in L^2_{p \times p}(0, \infty)$. Representation (5.49) follows from (5.57) and from the fact that $(z + i\delta)^{-2} \varphi(z + i\delta)$ is analytic. $\blacksquare$

Remark 5.5 *The integral representation (5.49) is true also for functions φ from the generalized Nevanlinna class $N_\varkappa$ under additional conditions*

$$\lim_{\eta \to \infty} \varphi(i\eta) = 0, \quad \lim_{\eta \to \infty} \eta |\Im \varphi(i\eta)| < \infty$$

in some semi-planes $\Im z > h_\varphi$ [28]. Weyl functions from the generalized Nevanlinna class appear, for instance, in [25, 26, 29, 34] (see also references therein).

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